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STOCHASTIC ANALYSIS BASED ON DETERMINISTIC BROWNIAN MOTION

BY

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ABSTRACT

A deterministic version of the Itô calculus is presented. We consider a model $Y_t = H(\mathbf{N}_t, t)$ with a deterministic Brownian $\mathbf{N}_t$ and an unknown function H. We predict Y_c from the observation $\{Y_t; t \in [a,b]\}$, where $a < b < c$. We prove that there exists an estimator Y_t based on the observation such that $E[(\hat{Y}_t - Y_c)^2] = O((c-b)^2)$ as $c \downarrow b$.

1. Introduction

Deterministic Brownian motions are stochastic processes with noncorrelated, stationary and strictly ergodic increments having 0-entropy and 0-expectation. The self-similarity of order 1/2 follows from these properties. Such processes have a lot of variety and have different properties. This is not the case of the Brownian motion where the process is characterized as a process with stationary and independent increments with 0-expectation and standard variance.

Among the deterministic Brownian motions, the simplest one is the N-process $(\mathbf{N}_t;\ t \in \mathbf{R})$ which is defined by the author in Example 8 of [K]. It comes from a piecewise linear function called the N_1-function (in Figure 1). It is time reversible.

The aim of this paper is to develop stochastic analysis based on the N-process. We consider a process $Y_t = H(\mathbf{N}_t, t)$, where the function $H(x,s)$ is twice continuously differentiable in x and once continuously differentiable in s and $H_x(x,s) > 0$. The function H is considered completely unknown except for these properties.

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We want to predict the value Y_c from the observation $Y_J := \{Y_t;\ t \in J\}$, where $J = [a,b]$ and $a < b < c$. We prove in Theorem 9 that there exists an estimator $\hat{Y}_c$ such that

$$\mathrm{E}[(\hat{Y}_c - Y_c)^2] = o((c-b)^2) + O\Big(\frac{(c-b)^2}{b-a}\Big) \tag{1}$$

as $c \downarrow b$ with the following $C(b)$ as the constant in $O(\ )$:

$$C(b) := 50 \sup_{|x| \le |b|^{1/2}} |H_x(x,b)|^2. \tag{2}$$

One of the motivations of our paper is given by Benoit B. Mandelbrot [2], who mentioned that the simulation of the stock market by the Brownian motion contains too much randomness. An actual market has a strong negative correlation between the fluctuations of price on a day and the next day. He suggests using the N-shaped function as the base of the simulation.

Our model has a lot of similarities to the Itô process. For example, we have an Itô formula (Theorem 4). Nevertheless, there is a big difference between them. Our process has 0-entropy while the Itô process has ∞-entropy. Therefore, we have a much better possibility of predicting the future. Theoretically, if we have complete information about the function H, and complete data of Y_t in the past, we should be able to predict the future without error. But the actual setting is with the unknown function H and the limited observation Y_t for a bounded interval J. The best we can do is order $O((c-b)^2)$ in the above estimate (1), and $O(c-b)$ in the case of an Itô process.

A sample path from an N-process repeats the N_1-function in various scales. The main idea for the prediction, called **synchronization**, is to find out the positions and the scales of the appearances of N_1-function in the sample path. An appearance of the N_1-function in a sample path is a part of bigger N_1-functions while containing smaller ones. Along the 3 line segments in an appearance of the N_1-function, the sample path either increases at the first part, then decreases and increases, or decreases at the first part, then increases and decreases. Thus, it has a strong correlation along the synchronized intervals, while the process itself has noncorrelated increments.

Another motivation is to create a sample path of Brownian motion in a deterministic way without using a random mechanism. Our N-process is strictly ergodic so that any chosen path realizes probablistic properties of the process. We don't need a randomization procedure but just take one, for example, the N_∞-function itself. Of course, it is not exactly like a path of the Brownian motion, but shares the quadratic structure with Brownian motion. If we take a derivative in some sense of the sample path, we get a white noise. Thus, our N-process provides a method of generating a random number.

2. N-process

We consider the **N-process** ($\mathbf{N}_t$; $t \in \mathbf{R}$), which is the stochastic process defined in Example 8 of [1] for $\alpha = 1/2$. We repeat the definition in a slightly different way as follows.

Define a continuous piecewise linear function N_1 (see Figure 1) on the interval $[0,1]$ by

$$N_1(x) = \begin{cases} \frac{3}{2}x, & 0 \le x \le 4/9, \\ -3x+2, & 4/9 \le x \le 5/9, \\ \frac{3}{2}x - \frac{1}{2}, & 5/9 \le x \le 1. \end{cases}$$

Let N_2 be the continuous piecewise linear function on $[0,1]$ obtained by replacing 3 line segments in N_1 by self-affine images of N_1 or $-N_1$ keeping the 2 end points fixed, that is,

$$N_2(x) = \begin{cases} \frac{2}{3}N_1(\frac{9}{4}x), & 0 \le x \le 4/9, \\ \frac{2}{3} - \frac{1}{3}N_1(9x-4), & 4/9 \le x \le 5/9, \\ \frac{1}{3} + \frac{2}{3}N_1(\frac{9}{4}x - \frac{5}{4}), & 5/9 \le x \le 1. \end{cases}$$

Let N_3 be the the continuous piecewise linear function on $[0,1]$ obtained by replacing 9 line segments in N_2 by self-affine images of N_1 or $-N_1$ as before. In the same way, we obtain N_n from N_{n-1} for $n = 4, 5, \ldots$. For covenience, we define N_0 by $N_0(t) = t$ for any $t \in [0,1]$.

We prove that the function N_n converges pointwise as n tends to infinity to a continuous function, say N_∞ on $[0,1]$. Let $a, b \in [0,1]$ with $a < b$. The interval $[a,b]$ is called a **synchronized interval of level** n if $\overline{(a, N_n(a))(b, N_n(b))}$ is one of the 3^n line segments consisting of the graph of the function N_n for $n = 0, 1, 2, \ldots$. In this case, we have for any $m \ge n$ that

1. $N_m(a) = N_n(a)$ and $N_m(b) = N_n(b)$,
2. $N_n(a) < N_m(t) < N_n(b)$ or $N_n(a) > N_m(t) > N_n(b)$ for any $t \in (a,b)$,
3. $|N_n(b) - N_n(a)| = |b-a|^{1/2}$, and
4. $b - a = \left(\frac{4}{9}\right)^i \left(\frac{1}{9}\right)^{n-i}$ for some $i = 0, 1, \ldots, n$.

Take any $t \in [0,1]$. For any $\varepsilon > 0$, there exists n and a synchronized interval of level n, say $[a,b]$ with $t \in [a,b]$ and $|b-a| < \varepsilon^2$. Then for any $m, m' \ge n$,

$$|N_m(t) - N_{m'}(t)| \le |N_n(b) - N_n(a)| = |b-a|^{1/2} < \varepsilon.$$

Thus, $N_m(t)$ converges as $m \to \infty$. The limit will be denoted by $N_\infty(t)$.

Let us prove the continuity of the function N_∞. Take any $s, t \in [0,1]$ with $0 < t - s \le (1/9)^n$ for some $n = 1, 2, \ldots$. Then there exists 2 neighboring synchronized intervals of level n, say $[a,b]$ and $[b,c]$ such that $[s,t] \subset [a,c]$. Then we have

$$|N_\infty(t) - N_\infty(s)| \le |N_n(b) - N_n(a)| + |N_n(c) - N_n(b)|$$
$$= |b-a|^{1/2} + |c-b|^{1/2} \le 2\left(\frac{4}{9}\right)^{n/2}$$

Thus, the function N_∞ is continuous.

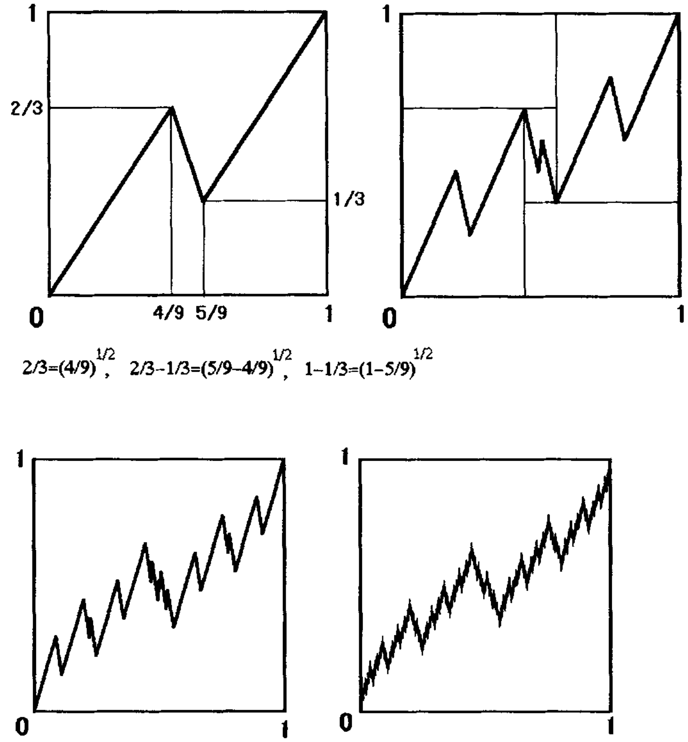

Figure 1. N_1, N_2, N_3 and N_∞.

We define a function $\tilde{N}_\infty : \mathbf{R} \to \mathbf{R}$ which is an extension of N_∞ by

$$\tilde{N}_\infty(t) = \begin{cases} 0, & t < 0, \\ N_\infty(t), & 0 \le t \le 1, \\ 1 & t > 1. \end{cases}$$

Now we randomize $\tilde{N}_\infty$ to get the N-process $(\mathbf{N}_t;\ t \in \mathbf{R})$.

Let Θ be the set of continuous functions $\omega\colon \mathbf{R} \to \mathbf{R}$ with $\omega(0) = 0$. We consider Θ as a topological space with the compact open topology, that is, $\omega_n \in \Theta$ converges to $\omega \in \Theta$ as n tends to infinity if and only if $\omega_n(t)$ converges to $\omega(t)$ uniformly on each bounded set of t. For $\omega \in \Theta$ and $s \in \mathbf{R}$, we define the **addition** $\omega + s \in \Theta$ (see Figure 2) by

$$(\omega + s)(t) = \omega(s + t) - \omega(s).$$

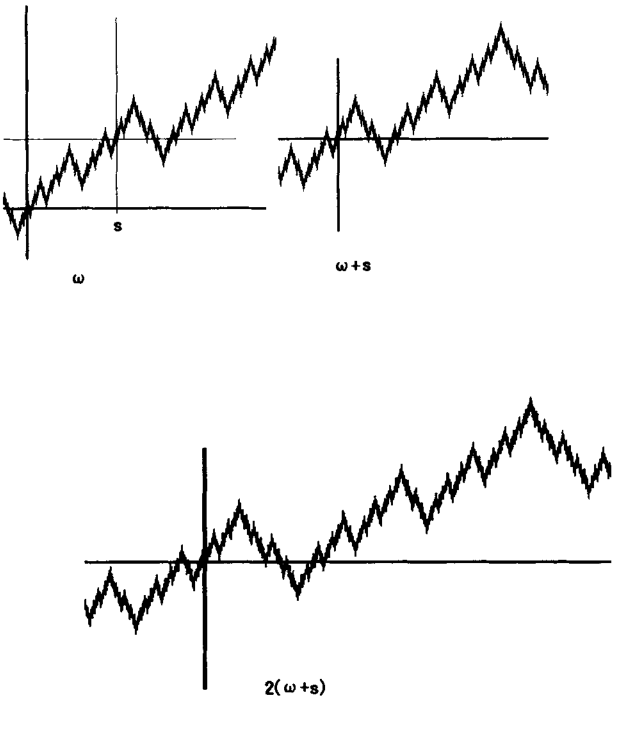

Figure 2. ω, $\omega + s$ and $2(\omega + s)$.

For $\omega \in \Theta$ and $\lambda \in \mathbf{R}_+$, we define the **multiplication** $\lambda\omega \in \Theta$ by

$$(\lambda\omega)(t) = \lambda^{1/2}\omega(\lambda^{-1}t).$$

Choose $s \in [0,1]$ randomly according to the Lebesgue measure on $[0,1]$ and define $\tilde{N}_\infty + s$. Now take $L > 0$ and choose $\lambda \in [0, L]$ randomly according to the normalized Lebesgue measure on $[0, L]$ independently of s and define $e^\lambda(\tilde{N}_\infty + s)$.

Now let L tend to infinity. We prove in Theorem 1 that the distribution of the random variable $e^{\lambda}(\tilde{N}_\infty + s)$ on Θ converges weakly (i.e. in the weak* sense) as L tends to infinity. Let P be the limiting distribution on Θ. Then the stochastic process $(\mathbf{N}_t;\ t \in \mathbf{R})$ on the probability space (Θ, P) is defined by $\mathbf{N}_t(\omega) = \omega(t)$ for any $\omega \in \Theta$ and $t \in \mathbf{R}$, which is called the **N-process**. Let Θ_0 be the topological support of the measure P.

Let $[a,b]$ be a synchronized interval of level i. We call it **increasing** if $N_\infty(a) < N_\infty(b)$ and **decreasing** if $N_\infty(a) > N_\infty(b)$. We call it **left, middle** or **right** if there exists a synchronized interval $[u,v]$ such that $[a,b]$ is equal to $[u,u']$, $[u',v']$ or $[v',v]$, respectively, where we put $u' = (5u+4v)/9$ and $v' = (4u+5v)/9$. For example, $[0,1]$ is the only synchronized interval of level 0, which is increasing. There are 3 synchronized intervals of level 1, namely $[0,4/9]$, $[4/9,5/9]$, $[5/9,1]$, which are increasing, decreasing and increasing, respectively and left, middle and right, respectively.

Let $X = e^{\lambda}(\tilde{N}_\infty + s)$ for some $s \in [0,1]$ and $\lambda \in [0,\infty)$. Note that

$$\begin{aligned} e^{\lambda} &= (X(\infty) - X(-\infty))^2, \\ 1 - s &= e^{-\lambda} \min\{t;\ X(t) = X(\infty)\}, \end{aligned}$$

so that λ and s are determined by X. Let $[a,b]$ be a synchronized interval. Then we say that $[(a-s)e^{\lambda}, (b-s)e^{\lambda}]$ is a **synchronized interval of** X. We also say that it is increasing, decreasing, left, middle or right synchronized interval of X if $[a,b]$ is so.

LEMMA 1: (1) *$\tilde{N}_\infty(t) + \tilde{N}_\infty(1-t) = 1$ for any $t \in \mathbf{R}$.*

(2) *Let $[a,b]$ be a synchronized interval. Then we have*

$$N_\infty(t) - N_\infty(a) = \xi(b-a)^{1/2} N_\infty\left(\frac{t-a}{b-a}\right)$$

for any $t \in [a,b]$, where ξ is 1 or -1 according as the interval $[a,b]$ is increasing or decreasing, respectively.

(3) *There exists a constant C such that*

$$|\tilde{N}_\infty(t) - \tilde{N}_\infty(s)| \le C|t-s|^{1/2}$$

for any $s,t \in \mathbf{R}$.

(4) *The set $K := \{e^{\lambda}(\tilde{N}_\infty + s);\ s \in [0,1],\ \lambda > 0\}$ is relatively compact in Θ.*

Remark 1: In Theorem 2, we prove that C in (3) of Lemma 1 can be taken as 1.

Proof: (1) Clear from the definitions of N_∞ and $\tilde{N}_\infty$.

(2) The graph of N_∞ restricted to the interval $[a,b]$ is the image of the graph of N_∞ by the affine transformation sending the point $(0,0)$ to $(a, N_\infty(a))$, $(0,1)$ to $(a, N_\infty(b))$, $(1,0)$ to $(b, N_\infty(a))$, and $(1,1)$ to $(b, N_\infty(b))$. Moreover, we already remarked that $N_\infty(b) - N_\infty(a) = \xi(b-a)^{1/2}$. Our conclusion follows from these properties.

(3) Assume without loss of generality that $0 \le s < t \le 1$ and $t - s < 1/2$, since otherwise, either the required inequality holds with $C = 2$ or it follows from our case by the symmetry or with $s \vee 0$ for s and $t \wedge 1$ for t. Take the maximum n such that there exist either 2 neighboring synchronized intervals $[a,b]$ and $[b,c]$ of level n with $[s,t] \subset [a,c]$. Then we have $t - s > (1/9)((b-a) \wedge (c-b))$, since otherwise, we can take a larger n than this. It follows that

$$\begin{aligned}
|\tilde{N}_\infty(t) - \tilde{N}_\infty(s)| &= |N_\infty(t) - N_\infty(s)| \\
&\le |N_\infty(b) - N_\infty(a)| + |N_\infty(c) - N_\infty(b)| \\
&= |b-a|^{1/2} + |c-b|^{1/2} \\
&= 3((b-a) \wedge (c-b))^{1/2} \\
&< 9|t-s|^{1/2},
\end{aligned}$$

where we used the fact that either $c - b = 4(b-a)$ or $c - b = (1/4)(b-a)$ holds, since $[a,b]$ and $[c,d]$ are neighboring synchronized intervals of the same level (see (2) of Lemma 2).

(4) By (3), any function f in K satisfies $|f(t) - f(s)| \le C|t-s|^{1/2}$ for any $s, t \in \mathbf{R}$ together with $f(0) = 0$. This implies that K is relatively compact in Θ. ∎

THEOREM 1: *The* N*-process introduced above is well defined and has the same distribution as the cocycle F for $\alpha = 1/2$ in* Example 8 *in* [1].

Proof: In Example 6 of [1], the weighted substitution (φ, η) on $\{0,1\}$ was defined as

$$\begin{aligned}
0 &\to \left(0, \frac{4}{9}\right)\left(1, \frac{1}{9}\right)\left(0, \frac{4}{9}\right), \\
1 &\to \left(1, \frac{4}{9}\right)\left(0, \frac{1}{9}\right)\left(1, \frac{4}{9}\right).
\end{aligned}$$

Then we defined $\Omega := \Omega(\varphi, \eta)$, the set of colored tilings associated to (φ, η) which is strictly ergodic with respect to the addition ($\mathbf{R}$-action). Let μ be the unique invariant measure on Ω with respect to the addition, which is also invariant under the multiplication ($\mathbf{R}_+$-action). Finally, we defined the 1/2-homogeneous cocycle F on Ω in Example 8 of [1]. Then

$$F(\omega,t)-F(\omega,c)=(-1)^{\sigma}(d-c)^{1/2}N_{\infty}\Big(\frac{t-c}{d-c}\Big) \tag{3}$$

for any $\omega\in\Omega$ and $t\in[c,d]$ if there exists a tile S of ω with color σ such that $S=(a,b]\times[c,d)$ for some a,b. For $\omega\in\Omega$, let $F(\omega)$ denote the function $\mathbf{R}\to\mathbf{R}$ such that $F(\omega)(t)=F(\omega,t)$. Then, $F(\omega)\in\Theta$. Let μ_F be the distribution of the random variable $F(\omega)$ with values in Θ defined on the probability space (Ω,μ).

We want to prove that the process $(\mathbf{N}_t;\ t\in\mathbf{R})$ is well defined and has the distribution μ_F. For this purpose, we prove that the distribution of the random variable $X_L:=e^{\lambda}(\tilde N_\infty+s)$ converges in the weak sense to μ_F as $L\to\infty$, where (s,λ) is a uniformly distributed random variable on $[0,1]\times[0,L]$. It is sufficient to prove that for any sequence $\{L_n;\ n=1,2,\ldots\}$ with $\lim_{n\to\infty}L_n=\infty$, there exists a subsequence $\{L_n'\}$ of $\{L_n\}$ with $\lim_{n\to\infty}L_n'=\infty$ such that the distribution of $X_{L_n'}$ converges to μ_F weakly as n tends to infinity.

Take any sequence $\{L_n;\ n=1,2,\ldots\}$ with $\lim_{n\to\infty}L_n=\infty$. There exists a subsequence $\{L_n'\}$ of $\{L_n\}$ with $\lim_{n\to\infty}L_n'=\infty$ such that the distribution of $X_{L_n'}$ converges weakly to, say, P', as n tends to infinity by (4) of Lemma 1. We want to prove that $P'=\mu_F$.

Since Ω is strictly ergodic with respect to the addition ([1]) and the transformation $F\colon\Omega\to\Theta$ is continuous satisfying $F(\omega+t)=F(\omega)+t$ ($\forall\omega\in\Omega$, $\forall t\in\mathbf{R}$), $F(\Omega)$ is strictly ergodic with respect to the addition. Hence it is sufficient to prove that

(i) P' is invariant under the addition, and

(ii) $P'(F(\Omega))=1$.

Let $\mathbf{L}$ be any bounded continuous funtional on Θ. Take any $t\in\mathbf{R}$ and $\eta\in\mathbf{R}_+$. Then we have

$$\begin{aligned}\int\mathbf{L}(\omega+t)dP'(\omega)&=\lim_{n\to\infty}\frac{1}{L_n'}\int_0^{L_n'}\int_0^1\mathbf{L}(e^{\lambda}(\tilde N_\infty+s)+t))dsd\lambda\\&=\lim_{n\to\infty}\frac{1}{L_n'}\int_0^{L_n'}\int_{te^{-\lambda}}^{1+te^{-\lambda}}\mathbf{L}(e^{\lambda}(\tilde N_\infty+s))dsd\lambda\\&=\int\mathbf{L}(\omega)dP'(\omega),\end{aligned}$$

which proves (i).

Since $F(\Omega)$ is compact ([1]), to prove (ii) it is sufficient to prove that $P'(F(\Omega)_M)=1$ for any $M>0$, where $F(\Omega)_M$ is the set of $f\in\Theta$ such that there exists $\omega\in\Omega$ satisfying that the restrictions of f and $F(\omega)$ to $[-M,M]$ coincide.

Let $[a_L, b_L]$ be the minimal synchronized interval of X_L, if it exists, containing $[-M, M]$ and let $c_L = 0$ or 1 corresponding to whether $[a_L, b_L]$ is increasing or decreasing. Such an interval $[a_L, b_L]$ exists if and only if

$$[-M, M] \subset [-se^{\lambda}, (1-s)e^{\lambda}], \tag{4}$$

since $[-se^{\lambda}, (1-s)e^{\lambda}]$ is the unique synchronized interval of X of level 0. In this case, take $\omega \in \Omega$ such that there exists a tile S of ω with color c_L and $S = (a, b] \times [a_L, b_L)$ for some a, b. Then by Lemma 1 and (3), we have

$$F(\omega, t) - F(\omega, a_L) = X_L(t) - X_L(a_L)$$

for any $t \in [-M, M] \subset [a_L, b_L]$. Since $F(\omega, 0) = X_L(0) = 0$, we have $F(\omega, a_L) = X_L(a_L)$ by putting $t = 0$ in the above equality. Hence, we have $F(\omega, t) = X_L(t)$ for any $t \in [-M, M]$. Thus,

$$X_L \in F(\Omega)_M \tag{5}$$

if (4) holds.

Let us estimate the probability that (4) holds.

$$\begin{aligned}\Pr([-M, M] \subset [-se^{\lambda}, (1-s)e^{\lambda}]) &= \Pr((s \wedge (1-s))e^{\lambda} \geq M) \\ &= \frac{1}{L}\int_0^L \int_0^1 1_{(s\wedge(1-s))e^{\lambda} \geq M} ds d\lambda \\ &\geq \frac{1}{L}\int_0^L (1 - 2Me^{-\lambda}) d\lambda \\ &\geq 1 - \frac{2M}{L}, \end{aligned} \tag{6}$$

which tends to 1 as L tends to infinity.

Since $F(\Omega)_M$ is a closed set we have, by (5) and (6),

$$P'(F(\Omega)_M)) \geq \lim_{n\to\infty} \Pr(X_{L'_n} \in F(\Omega)_M) = 1,$$

which proves (ii). ∎

Corollary 1: *The following statements hold.*

(1) *$\Theta_0 = F(\Omega)$, where Θ_0 is the topological support of the measure P.*

(2) *For any $\theta \in \Theta_0$ and $a, b \in \mathbf{R}$ with $a < b$, there exist $s \in [0, 1]$ and $\lambda \in [0, \infty)$ such that the restriction of θ to the interval $[a, b]$ coincides with the restriction of $e^{\lambda}(\tilde{N}_{\infty} + s)$ to $[a, b]$. Moreover, in this case, $[a, b] \subset [-se^{\lambda}, (1-s)e^{\lambda}]$ holds.*

COROLLARY 2 ([1]): *The space Θ_0 is compact and invariant under the addition and multiplication. The addition on Θ_0 is strictly ergodic with the unique invariant probability Borel measure P. Moreover, P is invariant under the multiplication. The entropy of the addition is 0. The stochastic process $(\mathbf{N}_t;\ t \in \mathbf{R})$ is self-similar with order 1/2 and has stationary, strictly ergodic and noncorrelated increments with 0 entropy. Moreover, $\mathrm{E}[\mathbf{N}_t] = 0$ and $\mathrm{V}[\mathbf{N}_t] = C|t|$ for any $t \in \mathbf{R}$, where $C > 0$ is a constant. Furthermore, the process $(\mathbf{N}_t;\ t \in \mathbf{R})$ is time reversible.*

Remark 2: We do not know the exact value of C in Corollary 2. A numerical computation tells us that $C = 0.1243\cdots$.

3. Synchronization

LEMMA 2: (1) *For any synchronized intervals I and J, either $I \subset J$, $I \supset J$ or $I^i \cap J^i = \emptyset$ holds, where I^i and J^i are the sets of interior points of I and J, respectively.*

(2) *For any neighboring synchronized intervals $[a, b]$ and $[b, c]$, either $(c-b)/(b-a) = (1/4)(4/9)^i$ for some integer i, or $(c-b)/(b-a) = 4(4/9)^i$ for some integer i, where i is the level of $[b, c]$ relative to $[a, b]$. Moreover, one of them is increasing and the other decreasing.*

Proof: (1) Clear from our construction of the function N_∞.

(2) Let $[u, v]$ be the minimal synchronized interval containing $[a, b] \cup [b, c]$ and let $[u, u']$, $[u', v']$, $[v', v]$ be the synchronized intervals of the next level, where $u' = (5u + 4v)/9$, $v' = (4u + 5v)/9$. Then, there are 2 cases:

CASE 1: $[a, b] \subset [u, u']$ and $[b, c] \subset [u', v']$.
In this case, we have $b - a = (4/9)^h(4/9)(v - u)$ and $c - b = (4/9)^k(1/9)(v - u)$, so that $(c - b)/(b - a) = (1/4)(4/9)^i$ with $i := k - h$, which is the level of $[b, c]$ relative to $[a, b]$.

CASE 2: $[a, b] \subset [u', v']$ and $[b, c] \subset [v', v]$.
In this case, we have $b - a = (4/9)^h(1/9)(v - u)$ and $c - b = (4/9)^k(4/9)(v - u)$, so that $(c-b)/(b-a) = 4(4/9)^i$ with $i := k-h$, which is the level of $[b, c]$ relative to $[a, b]$. ∎

LEMMA 3: *For any increasing (decreasing) synchronized interval $[a, b]$, we have $N_\infty(a) < N_\infty(t) < N_\infty(b)$ ($N_\infty(a) > N_\infty(t) > N_\infty(b)$, respectively) for any $t \in (a, b)$. In particular, $0 \le \tilde{N}_\infty(t) \le 1$ for any $t \in \mathbf{R}$.*

Proof: Let $[a, b]$ be an increasing synchronized interval of level n. Then, we remarked in Section 2 that $N_n(a) < N_m(t) < N_n(b)$ or $N_n(a) > N_m(t) > N_n(b)$ for any $t \in (a, b)$ and $m \geq n$. Since $N_n(a) = N_\infty(a) < N_\infty(b) = N_n(b)$, we have $N_\infty(a) < N_m(t) < N_\infty(b)$ for any $t \in (a, b)$ and $m \geq n$. Take any $t \in (a, b)$. There exists $m \geq n$ and a synchronized interval $[c, d]$ of level m such that $a < c \leq t \leq d < b$. Then,

$$N_\infty(a) < N_\infty(c) = N_m(c) \leq N_M(t) \leq N_m(d) = N_\infty(d) < N_\infty(a)$$

for any $M \geq m$. Letting $M \to \infty$, we have

$$N_\infty(a) < N_\infty(t) < N_\infty(a). \qquad ∎$$

LEMMA 4: (1) *For any* $0 < t \leq 1$, *we have* $N_\infty(t) \leq t^{1/2}$. *The equality holds if and only if* $[0, t]$ *is a synchronized interval.*

(2) *For any* $0 \leq t < 1$, $1 - N_\infty(t) \leq (1-t)^{1/2}$. *The equality holds if and only if* $[t, 1]$ *is a synchronized interval.*

Proof: (1) If $t \in (4/9, 5/9]$, then by Lemma 3,

$$N_\infty(t)/t^{1/2} < N_\infty(4/9)/(4/9)^{1/2} = 1.$$

Let

$$a = \frac{5}{9} + \left(\frac{4}{9}\right)^3 = \frac{469}{729}, \quad b = \frac{5}{9} + \left(\frac{4}{9}\right)^2 \frac{5}{9} = \frac{485}{729},$$
$$c = \frac{5}{9} + \left(\frac{4}{9}\right)^2 = \frac{61}{81}, \quad d = 1 - \left(\frac{4}{9}\right)^2 = \frac{65}{81}.$$

Then we have

$$N_\infty(a) = \frac{1}{3} + \left(\frac{2}{3}\right)^3 = \frac{17}{27} = \max_{5/9 \leq t \leq b} N_\infty(t),$$
$$N_\infty(c) = \frac{1}{3} + \left(\frac{2}{3}\right)^2 = \frac{7}{9} = \max_{b \leq t \leq d} N_\infty(t).$$

Hence,

$$N_\infty(t)/t^{1/2} < N_\infty(a)/(5/9)^{1/2} = \frac{17/27}{(5/9)^{1/2}} < 1$$

for any $t \in (5/9, b]$, and

$$N_\infty(t)/t^{1/2} < N_\infty(c)/b^{1/2} = \frac{7/9}{(485/729)^{1/2}} < 1$$

for any $t \in (b, d]$. If $t \in (d, 1)$, then there exists $k = 2, 3, \ldots$ such that $1 - (4/9)^k < t \le 1 - (4/9)^{k+1}$, and

$$N_\infty(t)/t^{1/2} < N_\infty(1 - (5/9)(4/9)^k)/(1 - (4/9)^k)^{1/2} = \frac{1 - \frac{1}{3}(\frac{2}{3})^k}{(1 - (4/9)^k)^{1/2}} < 1.$$

Therefore, $N_\infty(t)/t^{1/2} \ge 1$ holds only if $t = 1$ or $t \in (0, 4/9]$. For $t \in (0, 4/9]$, let $k = 1, 2, \ldots$ be such that $(4/9)^{k+1} < t \le (4/9)^k$. Then, since $[0, (4/9)^k]$ is a synchronized interval, we have by Lemma 1 that

$$N_\infty(t)/t^{1/2} = N_\infty((9/4)^k t)/((9/4)^k t)^{1/2}.$$

Since $(9/4)^k t \in (4/9, 1]$, $N_\infty(t)/t^{1/2} \ge 1$ if and only if $(9/4)^k t = 1$. That is, $t = (4/9)^k$. This is equivalent to saying that $[0, t]$ is a synchronized interval. Moreover, since the value of $N_\infty(t)/t^{1/2}$ at such t is 1, we complete the proof of (1).

(2) follows from (1) by (1) of Lemma 1. ∎

LEMMA 5: *For any $a, b \in \mathbf{R}$ with $a < b$, $|\tilde{N}_\infty(b) - \tilde{N}_\infty(a)| \le (b - a)^{1/2}$. The equality holds if and only if $[a, b]$ is a synchronized interval.*

Proof: If $a < b < 0$ or $1 < a < b$, then $|\tilde{N}_\infty(b) - \tilde{N}_\infty(a)| = 0 < (b - a)^{1/2}$. If $a < 0 < 1 < b$, then $|\tilde{N}_\infty(b) - \tilde{N}_\infty(a)| = 1 < (b - a)^{1/2}$. If $a < 0 \le b \le 1$, then $|\tilde{N}_\infty(b) - \tilde{N}_\infty(a)| = \tilde{N}_\infty(b) \le b^{1/2} < (b - a)^{1/2}$ by Lemma 4. If $0 \le a \le 1 < b$, then $|\tilde{N}_\infty(b) - \tilde{N}_\infty(a)| = 1 - \tilde{N}_\infty(a) \le (1 - a)^{1/2} < (b - a)^{1/2}$ by Lemma 4.

Finally, assume that $0 \le a < b \le 1$ and $\tilde{N}_\infty(a) = N_\infty(a)$, $\tilde{N}_\infty(b) = N_\infty(b)$. Let $[c, d]$ be the minimal synchronized interval containing $[a, b]$. We assume without loss of generality that the interval $[c, d]$ is increasing. Let $c' = (5c + 4d)/9$ and $d' = (4c + 5d)/9$. Then, the intervals $[c, c']$, $[c', d']$, $[d', d]$ are synchronized. By the assumption, $[a, b]$ is not contained in any of these intervals. Hence, there are 3 cases:

CASE 1: $a < c' < b \le d'$,

CASE 2: $c' \le a < d' < b$, and

CASE 3: $a < c' < d' < b$.

In Case 1, by Lemmas 1, 3 and 4, we have

$$\begin{aligned}|N_\infty(b)-N_\infty(a)| &\le (N_\infty(c')-N_\infty(a)) \vee (N_\infty(c')-N_\infty(b)) \\ &= (c'-c)^{1/2}N_\infty\Big(\frac{c'-a}{c'-c}\Big) \vee (d'-c')^{1/2}N_\infty\Big(\frac{b-c'}{d'-c'}\Big) \\ &\le (c'-c)^{1/2}\Big(\frac{c'-a}{c'-c}\Big)^{1/2} \vee (d'-c')^{1/2}\Big(\frac{b-c'}{d'-c'}\Big)^{1/2} \\ &= (c'-a)^{1/2} \vee (b-c')^{1/2} \\ &< (b-a)^{1/2}.\end{aligned}$$

In Case 2, by Lemmas 1, 3 and 4, we have

$$\begin{aligned}|N_\infty(b)-N_\infty(a)| &\le (N_\infty(a)-N_\infty(d')) \vee (N_\infty(b)-N_\infty(d')) \\ &= (d'-c')^{1/2}N_\infty\left(\frac{d'-a}{d'-c'}\right) \vee (d-d')^{1/2}N_\infty\left(\frac{b-d'}{d-d'}\right) \\ &\le (d'-c')^{1/2}\left(\frac{d'-a}{d'-c'}\right)^{1/2} \vee (d-d')^{1/2}\left(\frac{b-d'}{d-d'}\right)^{1/2} \\ &= (d'-a)^{1/2} \vee (b-d')^{1/2} \\ &< (b-a)^{1/2}.\end{aligned}$$

Let us consider Case 3. Let $A := N_\infty(c') - N_\infty(a)$ and $B := N_\infty(b) - N_\infty(d')$. Then we have $A > 0$ and $B > 0$ by Lemma 3. By Lemmas 1 and 4, we have $A^2 \le c' - a$ and $B^2 \le b - d'$. Moreover, $N_\infty(d') - N_\infty(c') = -(d'-c')^{1/2}$. Hence,

$$\begin{aligned}(N_\infty(b)-N_\infty(a))^2 &= (A+B-(d'-c')^{1/2})^2 \\ &= A^2+B^2+(d'-c')+2AB-2(A+B)(d'-c')^{1/2} \\ &\le b-a+2AB-2(A+B)(d'-c')^{1/2}. \qquad (7)\end{aligned}$$

Since $A \le (c'-c)^{1/2} = 2(d'-c')^{1/2}$ and $B \le (d-d')^{1/2} = 2(d'-c')^{1/2}$, we have

$$\begin{aligned}&2AB-2(A+B)(d'-c')^{1/2} \\ \le&2(d'-c')^{1/2}\cdot B + A\cdot 2(d'-c')^{1/2} - 2(A+B)(d'-c')^{1/2} = 0\end{aligned}$$

with equality only if $A = (c'-c)^{1/2}$ and $B = (d-d')^{1/2}$. Therefore by (7), we have $|N_\infty(b) - N_\infty(a)| \le (b-a)^{1/2}$ with equality only if $a = c$ and $b = d$ and the interval $[a, b]$ is synchronized. ∎

LEMMA 6: *Let $s \in [0,1]$ and $\lambda \in [0,\infty)$ be arbitrary and let $X := e^\lambda(\tilde N_\infty + s)$.*

(1) *For any interval $[a,b]$ $(a < b)$, we have $|X(b) - X(a)| \le (b-a)^{1/2}$.*

(2) *The following statements for an interval* $[a, b]$ $(a < b)$ *are equivalent to each other.*

(i) $[a, b]$ *is a synchronized interval of* X.

(ii) $X(a) \neq X(b)$ *and*

$$X(t) - X(a) = \xi(b-a)^{1/2}\tilde{N}_\infty\left(\frac{t-a}{b-a}\right)$$

for any $t \in [a, b]$, *where we set* $\xi := \mathrm{sgn}(X(b) - X(a))$.

(iii) $|X(b) - X(a)| = (b-a)^{1/2}$.

Proof: (1) follows from Lemma 5.

(2) It is clear that (ii) implies (iii). That (i) implies (ii) follows from Lemma 1. That (iii) implies (i) follows from Lemma 5. ∎

Let $\omega = (\mathbf{N}_t(\omega);\ t \in \mathbf{R})$ be an arbitrary sample path of the N-process belonging to Θ_0. Then by Corollary 1 its restriction to any bounded set is a restriction to the same set of some of X in Lemma 6. An interval $[a, b]$ $(a < b)$ is called a **synchronized interval of** ω if it is a synchronized interval of a function X as in Lemma 6 which coincides with ω on $[a - 4(b-a),\ b + 4(b-a)]$. This is well defined since it is independent of the choice of X by Lemma 6. It is called **increasing**, **decreasing**, **left**, **middle** or **right** if it is so in X as above. We cannot count the level of a synchronized interval of ω, but we can compare the levels between synchronized intervals. For two synchronized intervals I and J of ω, J is said to have **level** n $(n \in \mathbf{Z})$ **relative** to I if there exists X as in Lemma 6 which coincides with ω on an interval containing $I \cup J$ and $m \geq 0$ such that I and J are synchronized intervals of X with levels m and $m+n$, respectively. In particular, they are said to have the **same level** if $n = 0$ in the above. If two synchronized intervals I and J of ω satisfy $I \subset J$ and I has level n relative to J, we say that J is the n-th **ancester** of I.

THEOREM 2: *For any* $\omega \in \Theta_0$ *and an interval* $[a, b]$ $(a < b)$, $|\omega(b) - \omega(a)| \leq (b-a)^{1/2}$ *with equality if and only if* $[a, b]$ *is a synchronized interval of* ω. *If* $[a, b]$ *is a synchronized interval of* ω, *then*

$$\omega(t) - \omega(a) = \xi(b-a)^{1/2}N_\infty\left(\frac{t-a}{b-a}\right)$$

for any $t \in [a, b]$, *where we set* $\xi := \mathrm{sgn}(\omega(b) - \omega(a))$.

Proof: Clear from Lemma 6. ∎

LEMMA 7: *For any t with $0 < t \leq 1$, $N_\infty(t) \geq (1/3)t^{1/2}$.*

Proof: Take $k = 0, 1, 2, \ldots$ such that $(4/9)^{k+1} < t \leq (4/9)^k$. The minimum value of $N_\infty(s)$ for $(4/9)^{k+1} < s \leq (4/9)^k$ is $(1/3)(2/3)^k$, attained when $s = (5/9)(4/9)^k$. Therefore, we have

$$N_\infty(t) \geq (1/3)\Big(\frac{2}{3}\Big)^k = (1/3)\Big(\frac{4}{9}\Big)^{k/2} \geq (1/3)t^{1/2}. \qquad \blacksquare$$

For any $\omega \in \Theta_0$ and $\varepsilon > 0$, a closed interval I is called a $(1-\varepsilon)$-**synchronized interval** of ω if there exists a synchronized interval J of ω with $|I \cap J|/|I \cup J| \geq 1 - \varepsilon$.

THEOREM 3: *Let $\omega \in \Theta_0$. Then the following statements hold.*

(1) For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any interval $[a, b]$ $(a < b)$, if $|\omega(b) - \omega(a)| > (1-\delta)(b-a)^{1/2}$, then $[a, b]$ is an $(1-\varepsilon)$-synchronized interval of ω. In fact, for $\varepsilon < 1/10$, we can take $\delta = \varepsilon/18$.

(2) For any $\delta > 0$, there exists $\varepsilon > 0$ such that for any interval $[a, b]$ $(a < b)$, if $[a, b]$ is an $(1-\varepsilon)$-synchronized interval of ω, then $|\omega(b) - \omega(a)| > (1-\delta)(b-a)^{1/2}$. In fact, for $\delta < 1$, we can take $\varepsilon = (\delta/4)^2$.

(3) If $I = [u, v]$ is a $(1-\varepsilon)$-synchronized interval of ω with $0 < \varepsilon < 1/10$, then there exists a unique solution in u' and v' of the equation:

$$\begin{aligned} (8) \qquad & u',\ v' \in [u - (1/7)(v-u),\ v + (1/7)(v-u)], \\ & \omega(u') = \min\{\omega(t);\ t \in [u - (1/7)(v-u),\ v + (1/7)(v-u)]\}, \\ & \omega(v') = \max\{\omega(t);\ t \in [u - (1/7)(v-u),\ v + (1/7)(v-u)]\}. \end{aligned}$$

Let this solution be u', v'. Then the interval J defined as $J = [u', v']$ if $u' < v'$ and $J = [v', u']$ if $v' < u'$ is a synchronized interval of ω such that $|I \cap J|/|I \cup J| \geq 1 - \varepsilon$.

Proof: (1) Take any ε with $0 < \varepsilon < 1/20$. Assume that $[a, b]$ is not a $(1-2\varepsilon)$-synchronized interval of ω. Let $[c, d]$ be a minimal synchronized interval of ω containing $[a + \varepsilon(b-a),\ b - \varepsilon(b-a)]$. We assume without loss of generality that $[c, d]$ is increasing. Let $c' = (5c + 4d)/9$ and $d' = (4c + 5d)/9$. Then, by the minimality of $[c, d]$ and the assumption that $[a, b]$ is not $(1-2\varepsilon)$-synchronized, we have 6 cases.

CASE 1: $c - \varepsilon(b-a) \leq a \leq c + \varepsilon(b-a)$ and $c' + \varepsilon(b-a) < b \leq d'$.

CASE 2: $c - \varepsilon(b-a) \leq a \leq c + \varepsilon(b-a)$ and $d' < b < d - \varepsilon(b-a)$.

CASE 3: $c+\varepsilon(b-a) < a < c' - \varepsilon(b-a)$ and $c' + \varepsilon(b-a) < b \le d'$.

CASE 4: $c+\varepsilon(b-a) < a < c' - \varepsilon(b-a)$ and $d' < b \le d+\varepsilon(b-a)$.

CASE 5: $c' - \varepsilon(b-a) \le a \le c' + \varepsilon(b-a)$ and $d' + \varepsilon(b-a) < b \le d+\varepsilon(b-a)$.

CASE 6: $c' + \varepsilon(b-a) < a < d' - \varepsilon(b-a)$ and $d' + \varepsilon(b-a) < b \le d+\varepsilon(b-a)$.

In Case 1, by Theorem 2 and Lemma 7, we have

$$\begin{aligned}
|\omega(b)-\omega(a)| &= (\omega(c')-\omega(a)) - (\omega(c')-\omega(b)) \\
&\le (c'-a)^{1/2} - (d'-c')^{1/2} N_\infty\left(\frac{b-c'}{d'-c'}\right) \\
&\le (c'-a)^{1/2} - (d'-c')^{1/2}(1/3)\left(\frac{b-c'}{d'-c'}\right)^{1/2} \\
&= (c'-a)^{1/2} - (1/3)(b-c')^{1/2} \\
&\le (b-a)^{1/2} - (1/3)(\varepsilon(b-a))^{1/2} \\
&\le (b-a)^{1/2}(1-(\varepsilon/9)^{1/2}).
\end{aligned}$$

Hence, taking $\delta := (\varepsilon/9)^{1/2} > \varepsilon/9$ for 2ε, we have (1).

In Case 2, by Theorem 2 and Lemma 7, we have

$$\begin{aligned}
&(\omega(b)-\omega(a))^2 \\
&\quad = (A+B-C)^2 \\
&\quad \le (c'-a) + (b-d') + (d'-c') + 2AB - 2AC - 2BC \\
&\quad = b-a+2AB-(A+B)(2/3)(d-c)^{1/2} \\
&\quad \le b-a+(2/3)(d-c)^{1/2}B + AB - (A+B)(2/3)(d-c)^{1/2} \\
&\quad \le b-a-A(\omega(d)-\omega(d')-B) \\
&\quad = b-a-((\omega(c')-\omega(c))-(\omega(a)-\omega(c)))(\omega(d)-\omega(b)) \\
&\quad \le b-a-((2/3)(d-c)^{1/2}-|a-c|^{1/2})(1/3)(d-b)^{1/2} \\
&\quad \le b-a-\left((2/3)((1-2\varepsilon)(b-a))^{1/2}-(\varepsilon(b-a))^{1/2}\right)(1/3)(\varepsilon(b-a))^{1/2} \\
&\quad \le b-a-(1/12)\varepsilon^{1/2}(b-a) \\
&\quad \le (1-\varepsilon^{1/2}/12)(b-a),
\end{aligned}$$

where we put $A := \omega(c') - \omega(a)$, $B := \omega(b) - \omega(d')$ and $C := \omega(c') - \omega(d')$. Hence, taking $\delta := \varepsilon^{1/2}/12 > \varepsilon/9$ for 2ε, we have (1).

For Case 3, by Theorem 2 and Lemma 7, we have

$$\begin{aligned}(\omega(b)-\omega(a))^2 =&(A-B)^2\\ \leq&(c'-a)+(b-c')-2AB\\ \leq&b-a-2(1/3)(c'-a)^{1/2}(1/3)(b-c')^{1/2}\\ \leq&b-a-2((1/3)\varepsilon^{1/2}(b-a)^{1/2})^2\\ \leq&(1-(2\varepsilon/9))(b-a),\end{aligned}$$

where we put $A:=\omega(c')-\omega(a)$ and $B:=\omega(c')-\omega(b)$. Hence, taking $\delta:=2\varepsilon/9$ for 2ε, we have (1).

In Case 4, if $d'<b\leq d'+\varepsilon(b-a)$, then there exists b' with $c'+\varepsilon(b-a)<b'<d'$ and $\omega(b')=\omega(b)$. Hence (1) follows from Case 3 since

$$\begin{aligned}|\omega(b)-\omega(a)| =&|\omega(b')-\omega(a)|\\ \leq&(1-(2\varepsilon/9))(b'-a)\\ \leq&(1-(2\varepsilon/9))(b-a).\end{aligned}$$

Now assume that $d'+\varepsilon(b-a)<b\leq d+\varepsilon(b-a)$. By Theorem 2 and Lemma 7, we have

$$\begin{aligned}(\omega(b)-\omega(a))^2 =&(A+B-C)^2\\ \leq&(c'-a)+(b-d')+(d'-c')+2AB-2AC-2BC\\ =&b-a+2AB-(A+B)(2/3)(d-c)^{1/2}\\ \leq&b-a+A(2/3)(d-c)^{1/2}+AB-(A+B)(2/3)(d-c)^{1/2}\\ \leq&b-a-(\omega(c')-\omega(c)-A)B\\ \leq&b-a-(1/3)(a-c)^{1/2}(1/3)(b-d')^{1/2}\\ \leq&b-a-(1/9)\varepsilon(b-a)\\ =&(1-(\varepsilon/9))(b-a),\end{aligned}$$

where we put $A:=\omega(c')-\omega(a)$, $B:=\omega(b)-\omega(d')$, and $C:=\omega(c')-\omega(d')$. Hence taking $\delta:=\varepsilon/9$ for 2ε, we have (1).

Case 5 and Case 6 follow from the previous cases by symmetry.

(2) Let $0<\varepsilon<1/10$ and let $[a,b]$ be a $(1-\varepsilon)$-synchronized interval. Then there exists a synchronized interval $[c,d]$ with $|a-c|<2\varepsilon(b-a)$ and $|b-d|<2\varepsilon(b-a)$.

Then by Theorem 2, we have

$$\begin{aligned}|\omega(b)-\omega(a)| \geq& |\omega(d)-\omega(c)|-|\omega(a)-\omega(c)|-|\omega(b)-\omega(d)| \\ \geq& (d-c)^{1/2}-|a-c|^{1/2}-|b-d|^{1/2} \\ \geq& (b-a-\varepsilon(b-a))^{1/2}-2(2\varepsilon(b-a))^{1/2} \\ \geq& (1-4\varepsilon^{1/2})(b-a)^{1/2}.\end{aligned}$$

Thus, for any δ with $0<\delta<1$, we have (2) by taking $\varepsilon=(\delta/4)^2$.

(3) Assume without loss of generality that $\omega(a)<\omega(b)$. Then there exists a synchronized interval $J=[u',v']$ such that $|I\cap J|/|I\cup J|\geq 1-\varepsilon$. Moreover, u',v' is the unique solution of equation (8). ∎

4. Stochastic integral

Let $L=L(\omega)$ be a measurable function of $\omega\in\Theta_0$ taking a value in positve integers. Let $\{\zeta_0<\zeta_1<\ldots\}$ be a finite or infinite sequence of measurable functions of $\omega\in\Theta_0$ such that $[\zeta_i,\zeta_{i+1}]$ is a synchronized interval of $\omega\in\Theta_0$ for any $i=0,1,\ldots$ and ζ_L is defined for any $\omega\in\Theta_0$. We call a sequence $\zeta:=\{\zeta_0<\zeta_1<\cdots<\zeta_L\}$ a **synchronized net**. If, for an interval I, $I\subset[\zeta_0,\zeta_L]$ holds for any $\omega\in\Theta_0$, we say that ζ **covers** I. We denote $\|\zeta\|:=\|\max_{0\leq i\leq L-1}(\zeta_{i+1}-\zeta_i)\|_\infty$. Let $\mathcal{C}$ be a sub-σ-field of the probability space (Θ_0,P). If the above L and $\zeta_{i\wedge L}$ $(i=0,1,\ldots)$ are measurable with respect to $\mathcal{C}$, then we say that ζ is **measurable** with respect to $\mathcal{C}$ or ζ is $\mathcal{C}$**-measurable**. If $\{Y\}$ is a set of measurable functions on the probability space (Θ_0,P), then we say that ζ is $\{Y\}$-measurable if it is measurable with respect to the σ-field generated by the functions in $\{Y\}$. Let $\zeta=\{\zeta_0<\zeta_1<\cdots<\zeta_L\}$ and $\eta=\{\eta_0<\eta_1<\cdots<\eta_K\}$ be synchronized nets. If for any $\omega\in\Theta_0$, $\zeta\subset\eta$ holds between the sets of values of functions in ζ and η, and if η is measurable with respect to ζ, we say that η is a **refinement** of ζ.

LEMMA 8: *Let J be a bounded closed interval with $J=[a,b]$ $(a<b)$. Then, for any bounded closed interval I with $I\subset J^i$ and $\varepsilon>0$, there exists a synchronized net ζ covering I with $\|\zeta\|<\varepsilon$ which is measurable with respect to $d\mathbf{N}|_J$, where $\alpha\mathbf{N}|_J:=\{\mathbf{N}_t-\mathbf{N}_s;s,t\in J\}$.*

Proof: We may assume that $\varepsilon>0$ is small enough so that $I\subset[a+2\varepsilon,b-2\varepsilon]$.

1ST STEP: Let $\{(u_n,v_n);\ n=1,2,\ldots\}$ be a countable dense subset of

$$\{(x,y);-\varepsilon/2<x<0<y<\varepsilon/2,\ \varepsilon/18\leq y-x<\varepsilon/2\}.$$

Since there exists an synchronized interval $[c, d]$ of ω containing $a+\varepsilon$ with $\varepsilon/18 \le d-c < \varepsilon/2$, for δ with $0 < \delta < 1/200$, there exists $n = 1, 2, \ldots$ such that

$$|\omega(a+\varepsilon+v_n) - \omega(a+\varepsilon+u_n)| > (1-\delta)(v_n - u_n)^{1/2}.$$

Take the minimum n as this and define $d\mathbf{N}|_J$-measurable functions $u := a+\varepsilon+u_n$ and $v := a+\varepsilon+v_n$. Then by Theorem 3, $[u, v]$ is $(1-\delta')$-synchronized interval of ω for some $\delta' < 1/10$. Let u' and v' be the unique solution of equation (8) in Theorem 3 for this $(1-\delta')$-synchronized interval $[u, v]$. Then the functions u' and v' of $\omega \in \Theta$ are measurable with respect to $d\mathbf{N}|_J$. We define $\zeta_0 = u'$, $\zeta_1 = v'$ if $u' < v'$ and $\zeta_0 = v'$, $\zeta_1 = u'$ if $v' < u'$.

2ND STEP: Assume that a sequence of $d\mathbf{N}|_J$-measurable functions $\zeta_0 < \zeta_1 < \cdots < \zeta_k$ is defined so that $\zeta_0 < a + 2\varepsilon$ and $[\zeta_{i-1}, \zeta_i]$ is a synchronized interval with $\zeta_i - \zeta_{i-1} < \varepsilon$ for any $i = 1, 2, \ldots, k$. This is done for $k = 1$ in the 1st step.

We add ζ_{k+1} to get a longer sequence with this properties. Take the minimum nonnegative integer i such that $4(4/9)^i(\zeta_k - \zeta_{k-1}) < \varepsilon$. Since $[\zeta_{k-1}, \zeta_k]$ is a synchronized interval, for exactly one of ξ in $\{1/4, 4\}$, $[\zeta_k, \zeta_k + \xi(4/9)^i(\zeta_k - \zeta_{k-1})]$ is a synchronized interval. Define $\zeta_{k+1} = \zeta_k + \xi(4/9)^i(\zeta_k - \zeta_{k-1})$ with this ξ. Since ξ can be chosen in a $d\mathbf{N}|_J$-measurable way by Theorem 2, ζ_{k+1} is measurable with respect to $d\mathbf{N}|_J$ such that $\zeta_{k+1} - \zeta_k < \varepsilon$.

FINAL STEP: We prove that we can continue this process until we get $\zeta_{L+1} > b - 2\varepsilon$. Then, $\zeta := \{\zeta_0 < \zeta_1 < \cdots < \zeta_L\}$ satisfies the required properties.

The only possible obstruction against this is that ζ_k converges to some point, say $\eta \le b - \varepsilon$ as $k \to \infty$. We prove that this is impossible. To the contrary, suppose that this is the case. Then, there exists K such that for any $k \ge K$, the i in the description of the 2nd step is chosen as $i = 0$, so that all synchronized intervals $[\zeta_k, \zeta_{k+1}]$ for $k = K, K+1, \ldots$ have the same level. All consecutive $2 \cdot 3^n$ synchronized intervals of the same level contain a synchronized interval of level $-n$ relative to them for any $n = 1, 2, \ldots$. A synchronized interval of level $-n$ relative to the synchronized interval $[\zeta_K, \zeta_{K+1}]$ has length at least $(9/4)^n(\zeta_{K+1} - \zeta_K)$. Therefore, $\zeta_{K+2\cdot 3^n} - \zeta_K \ge (9/4)^n(\zeta_{K+1} - \zeta_K)$, which is a contradiction since, letting $n \to \infty$, we have $\eta - \zeta_k$ in the left-hand side and ∞ in the right-hand side. ∎

Let $A(\omega, s)$ be a function on $\Theta_0 \times \mathbf{R}$ which is measurable in ω and continuous in s for any fixed ω. Then for any $a, b \in \mathbf{R}$ with $a < b$, we define a **stochastic**

integral $\int_a^b Ad\mathbf{N}_t$ as follows:

$$\int_a^b Ad\mathbf{N}_t := \lim_{\substack{\|\zeta\|\to 0\\ \zeta_0\to a\\ \zeta_L\to b}} \sum_{i=0}^{L-1} A(\omega,\zeta_i)(\mathbf{N}_{\zeta_{i+1}} - \mathbf{N}_{\zeta_i}) \tag{9}$$

if the limit in the right-hand side exists, where $\zeta = \{\zeta_0 < \zeta_1 < \cdots < \zeta_L\}$ is a synchronized net.

THEOREM 4: *Let $H(x,s)$ be a real valued function of $x, s \in \mathbf{R}$ which is twice continuously differentiable in x and once continuously differentiable in s. Then for any $a < b$, the stochastic integral $\int_a^b H_x(\mathbf{N}_t,t)d\mathbf{N}_t$ exists and is $(H_x)_J \vee d\mathbf{N}|_J$-measurable with $J = [a,b]$, where $(H_x)_J := \{H(\mathbf{N}_t,t); t \in J\}$. Moreover, the following formula holds:*

$$H(\mathbf{N}_b,b) - H(\mathbf{N}_a,a) = \int_a^b H_x(\mathbf{N}_t,t)d\mathbf{N}_t + \int_a^b (\frac{1}{2}H_{xx}(\mathbf{N}_t,t) + H_s(\mathbf{N}_t,t))dt. \tag{10}$$

Proof: The $(H_x)_J \vee d\mathbf{N}|_J$-measurability of the stochastic integral follows from Lemma 8 if it exists, by taking the limit $\zeta_0 \downarrow a$ and $\zeta_L \uparrow b$. Therefore, it suffices to prove the existence of the stochastic integral and formula (10). For a net $\zeta = \{\zeta_0 < \zeta_1 < \cdots < \zeta_L\}$, denote

$$B(\zeta) := \sum_{i=0}^{L-1} H_x(\mathbf{N}_{\zeta_i},\zeta_i)(\mathbf{N}_{\zeta_{i+1}} - \mathbf{N}_{\zeta_i}).$$

Then, by the Taylor expansion of H and the continuity of H, H_{xx} and H_s in (x,s) as well as the sample path $\mathbf{N}_t$ in t, as $\|\zeta\| \to 0$, $\zeta_0 \to a$ and $\zeta_L \to b$ we have

$$\begin{aligned}
&H(\mathbf{N}_b,b) - H(\mathbf{N}_a,a)\\
&=\sum_{i=0}^{L-1} \big(H(\mathbf{N}_{\zeta_{i+1}},\zeta_{i+1}) - H(\mathbf{N}_{\zeta_i},\zeta_i)\big) + o(1)\\
&=\sum_{i=0}^{L-1} \big(H_x(\mathbf{N}_{\zeta_i},\zeta_i)(\mathbf{N}_{\zeta_{i+1}} - \mathbf{N}_{\zeta_i}) + \frac{1}{2}H_{xx}(\mathbf{N}_{\zeta_i},\zeta_i)(\mathbf{N}_{\zeta_{i+1}} - \mathbf{N}_{\zeta_i})^2\\
&\qquad + H_t(\mathbf{N}_{\zeta_i},\zeta_i)(\zeta_{i+1} - \zeta_i) + o(\zeta_{i+1} - \zeta_i)\big) + o(1)\\
&=B(\zeta) + \sum_{i=0}^{L-1} \big(\frac{1}{2}H_{xx}(\mathbf{N}_{\zeta_i},\zeta_i) + H_t(\mathbf{N}_{\zeta_i},\zeta_i)\big)(\zeta_{i+1} - \zeta_i) + o(1)\\
&=B(\zeta) + \int_a^b \big(\frac{1}{2}H_{xx}(\mathbf{N}_t,t) + H_t(\mathbf{N}_t,t)\big)dt + o(1),
\end{aligned}$$

where we used the fact that $(\mathbf{N}_{\zeta_{i+1}} - \mathbf{N}_{\zeta_i})^2 = \zeta_{i+1} - \zeta_i$. Hence, $B(\zeta)$ converges. Thus, the stochastic integral exists and we have (10). ∎

5. Prediction

Let $H(x, s)$ be a real valued function of $x, s \in \mathbf{R}$ such that

(H1) H is twice continuously differentiable in x and once continuously differentiable in s, and

(H2) $H_x(x, s) > 0$ for any $x, s \in \mathbf{R}$.

We consider the stochastic process $Y_t = H(\mathbf{N}_t, t)$ $(t \in \mathbf{R})$. Our problem is to predict Y_t for $t \notin J$ from the observation $Y_J := \{Y_t;\ t \in J\}$, where J is a bounded closed interval with nonempty interior. The function H is considered to be unknown except for the property (H1) and (H2). All the measurable functions of the observation Y_J we construct in the following do not need any further knowledge on the unknown function H.

THEOREM 5: *For any $\omega \in \Theta_0$ and $t \in \mathbf{R}$,*

$$H_x(\mathbf{N}_t, t) = \limsup_{\substack{u,v \to t \\ u<v}} \frac{|Y_v - Y_u|}{(v-u)^{1/2}} .$$

Let t_1, t_2 with $t_1 < t_2$ tend to t, attaining the lim sup *in the right-hand side of the above equality. Let $t_1' = (5t_1 + 4t_2)/9$ and $t_2' = (4t_1 + 5t_2)/9$. Then,*

$$\begin{aligned} H_{xx}(\mathbf{N}_t, t) =& \frac{9}{4} \lim \frac{-Y_{t_1} + Y_{t_1'} + Y_{t_2'} - Y_{t_2}}{(t_2 - t_1)^{1/2}}, \\ H_s(\mathbf{N}_t, t) =& \frac{3}{8} \lim \frac{Y_{t_1} - 9Y_{t_1'} + 3Y_{t_2'} + 5Y_{t_2}}{(t_2 - t_1)^{1/2}}. \end{aligned}$$

Therefore, if $t \in J$, then those quantities $H_x(\mathbf{N}_t, t)$, $H_{xx}(\mathbf{N}_t, t)$ and $H_s(\mathbf{N}_t, t)$ are measurable functions of the observation Y_J.

Proof: Since, by the Taylor expansion of H, we have

$$\begin{aligned} Y_v - Y_u =& H(\mathbf{N}_v, v) - H(\mathbf{N}_u, u) \\ =& H_x(\mathbf{N}_t, t)(\mathbf{N}_v - \mathbf{N}_u) + \frac{1}{2} H_{xx}(\mathbf{N}_t, t)(\mathbf{N}_v - \mathbf{N}_u)^2 \\ &+ H_s(\mathbf{N}_t, t)(v-u) + o(v-u) \end{aligned}$$

as $u, v \to t$, by Theorem 2 and (H2), we have

$$\begin{aligned} \limsup_{\substack{u,v \to t \\ u<v}} \frac{|Y_v - Y_u|}{(v-u)^{1/2}} =& H_x(\mathbf{N}_t, t) \limsup_{\substack{u,v \to t \\ u<v}} \frac{|\mathbf{N}_v - \mathbf{N}_u|}{(v-u)^{1/2}} \\ =& H_x(\mathbf{N}_t, t). \end{aligned}$$

By Theorem 3, the lim sup is attained if and only if $u, v \to t$, so that $[u, v]$ is an $(1-\varepsilon)$-synchronized interval of ω with $\varepsilon \to 0$. Therefore, the interval $[t_1, t_2]$ as in the statement of our theorem satisfies this condition. Furthermore, since we can approximate the $(1-\varepsilon)$-synchronized interval $[t_1, t_2]$ by a synchronized interval close to it and approximate the following quantities for the former by those for the latter with small errors, we may assume that $[t_1, t_2]$ itself is synchronized. Consider the Taylor expansions for

$$H(\mathbf{N}_{t_2'}, t_2') - H(\mathbf{N}_{t_1'}, t_1'),$$
$$H(\mathbf{N}_{t_2}, t_2) - H(\mathbf{N}_{t_1'}, t_1'),$$
$$H(\mathbf{N}_{t_2}, t_2) - H(\mathbf{N}_{t_1}, t_1),$$

and using the relations

$$\begin{aligned} t_2' - t_1' &= (1/9)(t_2 - t_1), \\ t_2 - t_1' &= (5/9)(t_2 - t_1), \\ \mathbf{N}_{t_2'} - \mathbf{N}_{t_1'} &= -(1/3)\xi(t_2 - t_1)^{1/2}, \\ \mathbf{N}_{t_2} - \mathbf{N}_{t_1'} &= (1/3)\xi(t_2 - t_1)^{1/2}, \\ \mathbf{N}_{t_2} - \mathbf{N}_{t_1} &= \xi(t_2 - t_1)^{1/2}, \end{aligned}$$

where $\xi = \operatorname{sgn}(\mathbf{N}_{t_2} - \mathbf{N}_{t_1})$, we have

$$\begin{aligned} Y_{t_2'} - Y_{t_1'} = &-(1/3)\xi H_x(\mathbf{N}_t, t)(t_2 - t_1)^{1/2} + (1/18)H_{xx}(\mathbf{N}_t, t)(t_2 - t_1) \\ &+ (1/9)H_s(\mathbf{N}_t, t)(t_2 - t_1) + o(t_2 - t_1), \\ Y_{t_2} - Y_{t_1'} = &(1/3)\xi H_x(\mathbf{N}_t, t)(t_2 - t_1)^{1/2} + (1/18)H_{xx}(\mathbf{N}_t, t)(t_2 - t_1) \\ &+ (5/9)H_s(\mathbf{N}_t, t)(t_2 - t_1) + o(t_2 - t_1), \end{aligned}$$

and

$$\begin{aligned} Y_{t_2} - Y_{t_1} = &\xi H_x(\mathbf{N}_t, t)(t_2 - t_1)^{1/2} + \frac{1}{2}H_{xx}(\mathbf{N}_t, t)(t_2 - t_1) \\ &+ H_s(\mathbf{N}_t, t)(t_2 - t_1) + o(t_2 - t_1). \end{aligned}$$

By solving the above linear equation on $H_x(\mathbf{N}_t, t), H_{xx}(\mathbf{N}_t, t), H_s(\mathbf{N}_t, t)$ and letting $t_2 - t_1 \to 0$, we get the required formulas for $H_{xx}(\mathbf{N}_t, t)$ and $H_t(\mathbf{N}_t, t)$.

It is clear from the above formulas that if t belongs to the interior of J, then the quantities $H_{xx}(\mathbf{N}_t, t)$ and $H_t(\mathbf{N}_t, t)$ are measurable with respect to the observation Y_J. It follows from the continuity that the same result holds for any $t \in J$. ∎

THEOREM 6: *Let I, J be closed intervals with $J = [a,b]$ $(a < b)$ and $\emptyset \neq I^i \subset I \subset (a,b)$.*

(1) *For any $\delta > 0$, there exists $\varepsilon > 0$ such that for any $t \in J$ and $u, v \in (t-\varepsilon, t+\varepsilon)$,*

$$Y_v - Y_u = H_x(\mathbf{N}_t, t)(\mathbf{N}_v - \mathbf{N}_u) + \Xi$$

with

$$|\Xi| \leq \delta(|\mathbf{N}_v - \mathbf{N}_u| + |v-u|^{1/2}).$$

(2) *For any $\varepsilon > 0$, there exists a Y_J-measurable synchronized net covering I with $\| \zeta \| < \varepsilon$.*

(3) *$d\mathbf{N}|_J$ is measurable with respect to the observation Y_J. Hence, both terms in the right-hand side of (10) are Y_J-measurable.*

Proof: (1) For any given $\delta > 0$, take ε with $0 < \varepsilon < 1$ satisfying
(i) $|H_x(x', s') - H_x(x, s)| < \delta$ for any (x, s) and (x', s') with

$$s, s' \in J', \; |s - s'| < \varepsilon, \; |x|, |x'| \leq (|a'|^{\vee}|b'|)^{1/2} \text{ and } |x - x'| < \varepsilon^{1/2},$$

(ii) $\sup_{s \in J', \, |x| \leq (|a'|^{\vee}|b'|)^{1/2}} |H_s(x,s)| \cdot (2\varepsilon)^{1/2} < \delta$,
where $a' = a - 1$, $b' = b + 1$, $J' := [a', b']$. Then for any $t \in J$ and $u, v \in (t - \varepsilon, t + \varepsilon)$,

$$\begin{aligned} Y_v - Y_u =& H(\mathbf{N}_v, v) - H(\mathbf{N}_u, u) \\ =& (H(\mathbf{N}_v, v) - H(\mathbf{N}_v, u)) + (H(\mathbf{N}_v, u) - H(\mathbf{N}_u, u)) \\ =& H_s(\mathbf{N}_v, t')(v - u) + H_x(x', u)(\mathbf{N}_v - \mathbf{N}_u) \\ =& H_x(\mathbf{N}_t, t)(\mathbf{N}_v - \mathbf{N}_u) + \Xi \end{aligned}$$

with

$$\Xi := H_s(\mathbf{N}_v, t')(v-u) + (H_x(x', u) - H_x(\mathbf{N}_t, t))(\mathbf{N}_v - \mathbf{N}_u),$$

where t' and x' satisfy $|t' - t| < \varepsilon$ and $|x' - \mathbf{N}_t| < \varepsilon^{1/2}$. Then using (i) and (ii), we have

$$\begin{aligned} |\Xi| \leq& |H_s(\mathbf{N}_v, t')||v - u| + |H_x(x', u) - H_x(\mathbf{N}_t, t)||\mathbf{N}_v - \mathbf{N}_u| \\ \leq& |H_s(\mathbf{N}_v, t')|(2\varepsilon)^{1/2}|v-u|^{1/2} + \delta|\mathbf{N}_v - \mathbf{N}_u| \\ \leq& \delta(|\mathbf{N}_v - \mathbf{N}_u| + |v-u|^{1/2}). \end{aligned}$$

(2) Take sufficiently small $\delta > 0$ determined finally in the following 2nd step. At this moment, we assume that

$$(11) \qquad 0 < \delta < \inf_{t \in J, \; |x| \leq (|a|^{\vee}|b|)^{1/2}} H_x(x,t)/1200.$$

We may assume that $\varepsilon > 0$ is small enough so that the statement (1) holds with this δ and $I \subset [a+2\varepsilon, b-2\varepsilon]$. We use a similar construction as in the proof of Lemma 8.

1ST STEP: Let $\{(u_n, v_n);\ n = 1, 2, \ldots\}$ be a countable dense subset of

$$\{(x,y); -\varepsilon/2 < x < 0 < y < \varepsilon/2,\ \varepsilon/18 \le y - x < \varepsilon/2\}.$$

There exists a synchronized interval $[c,d]$ of ω containing $t := a + \varepsilon$ with $\varepsilon/18 \le d - c < \varepsilon/2$. Then, we have by (1)

$$\begin{aligned}|Y_d - Y_c| \ge& (H_x(\mathbf{N}_t, t) - \delta)|\mathbf{N}_d - \mathbf{N}_c| - \delta(d-c)^{1/2}\\ =&(H_x(\mathbf{N}_t, t) - \delta)(d-c)^{1/2} - \delta(d-c)^{1/2}\\ =&(H_x(\mathbf{N}_t, t) - 2\delta)(d-c)^{1/2}.\end{aligned}$$

Hence, there exists $n = 1, 2, \ldots$ such that

$$|Y_{t+v_n} - Y_{t+u_n}| > (H_x(\mathbf{N}_t, t) - 3\delta)(v_n - u_n)^{1/2}.$$

Take the minimum n as this and define functions $u := t + u_n$ and $v := t + v_n$, which are Y_J-measurable by Theorem 5.

Since as above we have

$$\begin{aligned}(H_x(\mathbf{N}_t, t) - 3\delta)(v-u)^{1/2} <& |Y_v - Y_u|\\ \le& (H_x(\mathbf{N}_t, t) + \delta)|\mathbf{N}_v - \mathbf{N}_u| + \delta(v-u)^{1/2},\end{aligned}$$

we have by (11) that

$$|\mathbf{N}_v - \mathbf{N}_u| > (1 - 1/200)(v-u)^{1/2}.$$

Then by Theorem 3, $[u,v]$ is a $(1-1/11)$-synchronized interval of ω. Let u' and v' be the unique solution of equation (8) in Theorem 3 for this $(1-1/11)$-synchronized interval $[u,v]$.

We prove that u', v' is also the unique solution of the equation

$$\begin{aligned}&u',\ v' \in [u - (1/7)(v-u),\ v + (1/7)(v-u)],\\ &Y_{u'} = \min\{Y_s;\ s \in [u - (1/7)(v-u),\ v + (1/7)(v-u)]\},\\ &Y_{v'} = \max\{Y_s;\ s \in [u - (1/7)(v-u),\ v + (1/7)(v-u)]\}.\end{aligned} \tag{12}$$

Take any $s \in [u - (1/7)(v-u),\ v + (1/7)(v-u)]$ with $s \neq u'$. Then by Lemma

7, $\mathbf{N}_s - \mathbf{N}_{u'} \geq (1/3)|s-u'|^{1/2}$. Therefore as above, we have

$$\begin{aligned} Y_s - Y_{u'} \geq & (H_x(t,\mathbf{N}_t) - \delta)(\mathbf{N}_s - \mathbf{N}_{u'}) - \delta|s-u'|^{1/2} \\ \geq & (H_x(\mathbf{N}_t,t) - \delta)(1/3)|s-u'|^{1/2} - \delta|s-u'|^{1/2} \\ = & (H_x(\mathbf{N}_t,t) - 4\delta)(1/3)|s-u'|^{1/2} \\ \geq & (1200-4)\delta(1/3)|s-u'|^{1/2}, \end{aligned}$$

so that u' is the unique solution of equation (12). Similarly, v' is the unique solution of equation (12). Thus, u' and v' are Y_J-measurable functions on $\omega \in \Theta$.

We define $\zeta_0 = u'$, $\zeta_1 = v'$ if $u' < v'$ and $\zeta_0 = v'$, $\zeta_1 = u'$ if $v' < u'$.

2ND STEP: Assume that a sequence of Y_J-measurable functions $\zeta_0 < \zeta_1 < \cdots < \zeta_k$ is defined so that $\zeta_0 < a + 2\varepsilon$ and $[\zeta_{i-1}, \zeta_i]$ is a synchronized interval with $\zeta_{i-1} - \zeta_i < \varepsilon$ for any $i = 1, 2, \ldots, k$. This is done for $k = 1$ in the 1st step.

We add ζ_{k+1} to get a longer sequence with these properties. Take the minimum nonnegative integer i such that $4(4/9)^i(\zeta_k - \zeta_{k-1}) < \varepsilon$. Since $[\zeta_{k-1}, \zeta_k]$ is a synchronized interval, for exactly one of ξ in $\{1/4, 4\}$, $[\zeta_k, \zeta_k + \xi(4/9)^i(\zeta_k - \zeta_{k-1})]$ is a synchronized interval. Define $\zeta_{k+1} = \zeta_k + \xi(4/9)^i(\zeta_k - \zeta_{k-1})$ with this ξ.

What we have to prove is that ξ is chosen in a Y_J-measurable way. Let $\xi \in \{1/4, 4\}$ be such that $[t, \zeta]$ is a synchronized interval and let $\xi' \in \{1/4, 4\}$ be $\xi' \neq \xi$, so that $[t, \zeta']$ is not a synchronized interval, where we put $t := \zeta_k$, $\zeta := t + \xi(4/9)^i(t - \zeta_{k-1})$ and $\zeta' = t + \xi'(4/9)^i(t - \zeta_{k-1})$. Let $[t, \zeta'']$ be the minimal synchronized interval containing $[t, \zeta']$. Then, we can prove that there exists $p > 0$ such that $(4/9) + p < (\zeta' - t)/(\zeta'' - t) < 1 - p$. Therefore, by Theorem 2, there exists q with $1/2 < q < 1$ such that

$$|\mathbf{N}_{\zeta'} - \mathbf{N}_t| < q|\zeta' - t|^{1/2}$$

while

$$|\mathbf{N}_{\zeta} - \mathbf{N}_t| = |\zeta - t|^{1/2}.$$

Then, as we proved in the 1st step, we have

$$\begin{aligned} |Y_{\zeta'} - Y_t| \leq & (H_x(\mathbf{N}_t,t) + \delta)|\mathbf{N}_{\zeta'} - \mathbf{N}_t| + \delta(\zeta' - t)^{1/2} \\ \leq & (H_x(\mathbf{N}_t,t) + 3\delta)q(\zeta' - t)^{1/2}, \end{aligned}$$

while

$$\begin{aligned} |Y_{\zeta} - Y_t| \geq & (H_x(\mathbf{N}_t,t) - \delta)|\mathbf{N}_{\zeta} - \mathbf{N}_t| - \delta(\zeta - t)^{1/2} \\ = & (H_x(\mathbf{N}_t,t) - 2\delta)(\zeta - t)^{1/2}. \end{aligned}$$

Therefore, by choosing small $\delta > 0$, we have

$$|Y_{\zeta'} - Y_t|/(\zeta' - t)^{1/2} \leq H_x(\mathbf{N}_t, t)(1 + 2q)/3,$$
$$|Y_\zeta - Y_t|/(\zeta - t)^{1/2} \geq H_x(\mathbf{N}_t, t)(2 + q)/3,$$

so that we can distinguish these 2 cases by the observation Y_J. Hence, ξ is Y_J-measurable.

Thus, the function ζ_{k+1} on $\omega \in \Theta$ is Y_J-measurable such that $[\zeta_k, \zeta_{k+1}]$ is a synchronized interval with $\zeta_{k+1} - \zeta_k < \varepsilon$.

FINAL STEP: We continue this process until we get $\zeta_{L+1} > b - \varepsilon$. Then, $\zeta := \{\zeta_0 < \zeta_1 < \cdots < \zeta_L\}$ satisfies the required properties. This can be done by the same reasoning as in the final step of the proof of Lemma 8.

(3) Let $\zeta = \{\zeta_0 < \zeta_1 < \cdots < \zeta_L\}$ be a Y_I-measurable synchronized net covering J. If necessary, we repeat the division of a synchronized interval $[\zeta_i, \zeta_{i+1}]$ by $[\zeta_i, \zeta'_i]$, $[\zeta'_i, \zeta'_{i+1}]$, $[\zeta'_{i+1}, \zeta_{i+1}]$ with $\zeta'_i = (5\zeta_i + 4\zeta_{i+1})/9$ and $\zeta'_{i+1} = (4\zeta_i + 5\zeta_{i+1})/9$; we may assume that there exists $[\zeta_i, \zeta_{i+1}] \subset I^i$ such that $\zeta_{i+1} - \zeta_i$ is sufficiently small so that $Y_{\zeta_{i+1}} - Y_{\zeta_i}$ has the same sign as $\mathbf{N}_{\zeta_{i+1}} - \mathbf{N}_{\zeta_i}$. Then, we know from the observation Y_I whether the synchronized interval $[\zeta_i, \zeta_{i+1}]$ is increasing or decreasing. Since the synchronized intervals $[\zeta_j, \zeta_{j+1}]$'s are increasing and decreasing alternatively, we know $\xi = \operatorname{sgn}(\mathbf{N}_{j+1} - \mathbf{N}_j)$ for all $j = 0, 1, \ldots, L-1$. Since

$$\mathbf{N}_t - \mathbf{N}_{\zeta_j} = \xi(t - \zeta_j)^{1/2} N_\infty\Big(\frac{t - \zeta_j}{\zeta_{j+1} - \zeta_j}\Big)$$

for any $t \in [\zeta_j, \zeta_{j+1}]$ by Theorem 2, we get $d\mathbf{N}|_J$ from the observation Y_I, hence by Y_J considering the limit. ∎

LEMMA 9: (1) *Let $\{\zeta_0 < \zeta_0 < \zeta_1 < \cdots < \zeta_L\}$ be a synchronized net. Let $(\zeta_{i+1} - \zeta_i)/(\zeta_i - \zeta_{i-1}) = \xi(4/9)^j$ with $\xi \in \{1/4, 4\}$ and $j \in \mathbf{Z}$ for some $i = 1, 2, \ldots, L-1$ and $\omega \in \Theta$. If $j > 0$, then for $\eta := \zeta_i + \xi(\zeta_i - \zeta_{i-1})$, $[\zeta_i,\ \eta]$ is a synchronized interval of $\omega \in \Theta_0$, and if $\eta \leq \zeta_L$, then there exists n with $i+1 < n \leq L$ such that $\eta = \zeta_n$. If $j < 0$, then for $\eta := \zeta_i - \xi(\zeta_{i+1} - \zeta_i)$, $[\eta,\ \zeta_i]$ is a synchronized interval of $\omega \in \Theta_0$, and if $\eta \geq \zeta_0$, then there exists n with $0 \leq n < i-1$ such that $\eta = \zeta_n$.*

(2) *For any neighboring synchronized intervals $[a, b]$, $[b, c]$ and $[c, d]$ of $\omega \in \Theta_0$, if $(c-b)/(b-a) = 1/4$ and $(d-c)/(c-b) = 4$, then $[a, d]$ is a synchronized interval of ω.*

(3) *For any neighboring synchronized intervals* $[a, b]$, $[b, c]$ *and* $[c, d]$ *of* $\omega \in \Theta_0$, *if* $(c-b)/(b-a) = 1/4$ *and* $(d-c)/(c-b) = 1/4$, *then* $[a-(9/4)(b-a), b]$ *and* $[b, b+(9/4)(c-b)]$ *are synchronized intervals of* ω.

(4) *For any neighboring synchronized intervals* $[a, b]$, $[b, c]$ *and* $[c, d]$ *of* $\omega \in \Theta_0$, *if* $(c-b)/(b-a) = 4$ *and* $(d-c)/(c-b) = 4$, *then* $[b-(9/4)(c-b), c]$ *and* $[c, c+(9/4)(d-c)]$ *are synchronized intervals of* ω.

Proof: (1) Assume that $j > 0$. Let K be the nearest common ancester of $[\zeta_{i-1}, \zeta_i]$ and $[\zeta_i, \zeta_{i+1}]$. Let $[\zeta_{i-1}, \zeta_i]$ have level k relative to K. Then by (2) of Lemma 2, $[\zeta_i, \zeta_{i+1}]$ has level $k+j$ relative to K. Since $k > 0$, the j-th ancester of $[\zeta_i, \zeta_{i+1}]$, is neighboring to $[\zeta_{i-1}, \zeta_i]$. Let it be $[\zeta_i, \eta]$. Then, $\eta - \zeta_i = \xi(\zeta_i - \zeta_{i-1})$. If $\eta \le \zeta_L$, then by (1) of Lemma 2, there exists n with $i+1 < n \le L$ such that $\eta = \zeta_n$. The proof for the case $j < 0$ is similar.

(2) Let K be the nearest common ancester of $[a, b]$, $[b, c]$ and $[c, d]$. It is sufficient to prove that $K = [a, d]$. Suppose to the contrary that $K \ne [a, d]$. Then, $[b, c]$ has level $j > 1$ relative to K and is not middle. Assume that it is left. Then, $[c, d]$ is middle since $[b, c]$ and $[c, d]$ have the same level. Thus $(d-c)/(c-b) = 1/4$, contradicting the assumption. If $[b, c]$ is right, we have $(c-b)/(b-a) = 4$, contradicting the assumption.

(3) Since neither $[a, b]$ nor $[b, c]$ is middle by the assumption, we have that $[a, b]$ is right and $[b, c]$ is left. Then, the first ancestor of $[a, b]$ is $[b-(9/4)(b-a),\ b]$ and the first ancestor of $[b, c]$ is $[b,\ b+(9/4)(c-b)]$.

(4) Let K be the nearest common ancester of $[a, b]$ and $[b, c]$. If K is not the first ancestor of $[a, b]$ and $[b, c]$, then $[b, c]$ is left, which contradicts $(d-c)/(c-b) = 4$. Hence, K is the first ancestor of $[a, b]$ and $[b, c]$. This implies that $K = [c-(9/4)(c-b),\ c]$ and that K is not an ancestor of $[c, d]$, since $(c-b)/(b-a) = 4$. Therefore, the nearest common ancestor of $[b, c]$ and $[c, d]$ is not their first ancestor. Thus, $[c, d]$ is left and the first ancestor of $[c, d]$ is $[c,\ c+(9/4)(d-c)]$. ∎

Let $\zeta = \{\zeta_0 < \zeta_1 < \cdots < \zeta_L\}$ and $\eta = \{\eta_0 < \eta_1 < \cdots < \eta_M\}$ be synchronized nets such that η is measurable with respect to ζ. We say that η is a **reduction** of ζ if $\eta_0 \le \zeta_0 < \zeta_L \le \eta_M$ and $\{\eta_1 < \eta_2 < \cdots < \eta_{M-1}\} \subset \{\zeta_1 < \zeta_2 < \cdots < \zeta_{L-1}\}$ holds.

THEOREM 7: *For any* Y_J*-measurable synchronized net* $\zeta = \{\zeta_0 < \zeta_1 < \cdots < \zeta_L\}$, *there exists a reduction of it consisting at most of 3 synchronized intervals with the same level.*

Proof: Let $\eta = \{\eta_0 < \eta_1 < \cdots < \eta_M\}$ be a reduction of ζ with the smallest number of intervals M. If the levels of the synchronized intervals contained in it are not the same, then there exists $i = 0, 1, \ldots, M-1$ such that

$$(\eta_{i+1} - \eta_i)/(\eta_i - \eta_{i-1}) = \xi(4/9)^j \quad \text{with } \xi \in \{1/4, 4\} \text{ and } j \neq 0.$$

If $j > 0$, then by Lemma 9, $[\eta_i,\ \eta_i + \xi(\eta_i - \eta_{i-1})]$ is a synchronized interval and either there exists n with $i+1 < n \leq M$ such that $\eta_n = \eta_i + \xi(\eta_i - \eta_{i-1})$ or $\eta_i + \xi(\eta_i - \eta_{i-1}) > \eta_L$. In the former case, we have a further reduction of ζ, $\{\eta_0 < \eta_1 < \cdots < \eta_i < \eta_n < \cdots < \eta_M\}$ with a number of intervals less than M, contradicting the assumption on M. In the latter case, we have a futher reduction of ζ, $\eta' := \{\eta_0 < \eta_1 < \cdots < \eta_i < \eta_i + \xi(\eta_i - \eta_{i-1})\}$, which has a number of intervals at most M. By the assumption on M, it is exactly M and $i = M-1$.

If $j < 0$, then by Lemma 9, $[\eta_i - \xi(\eta_{i+1} - \eta_i),\ \eta_i]$ is a synchronized interval and either there exists n with $0 \leq n < i-1$ such that $\eta_n = \eta_i - \xi(\eta_{i+1} - \eta_i)$ or $\eta_i - \xi(\eta_{i+1} - \eta_i) < \eta_0$. In the former case, we have a further reduction of ζ, $\{\eta_0 < \eta_1 < \cdots < \eta_n < \eta_i < \cdots < \eta_M\}$ with a number of intervals less than M, contradicting the assumption on M. In the latter case, we have a futher reduction of ζ, $\eta' := \{\eta_i - \xi(\eta_{i+1} - \eta_i) < \eta_i < \cdots < \eta_M\}$, which has a number of intervals at most M. By the assumption on M, it is exactly M and $i = 1$.

If the levels of the synchronized intervals contained in η' are not the same, we repeat the above procedure to get finally a futher reduction of ζ such that it has a number M of synchronized intervals with the same level. Hence, we may assume that $\eta = \{\eta_0 < \eta_1 < \cdots < \eta_M\}$ is a reduction of ζ which has the smallest number of intervals M with the same level.

Suppose that $M \geq 4$. Then, in the sequence of $(\eta_{i+1} - \eta_i)/(\eta_i - \eta_{i-1})$ $(i = 1, 2, \ldots, M-1)$, there exists $i = 1, 2, \ldots, M-2$ such that the combination $((\eta_{i+1} - \eta_i)/(\eta_i - \eta_{i-1}),\ (\eta_{i+2} - \eta_{i+1})/(\eta_{i+1} - \eta_i))$ is either $(1/4,\ 4)$, $(1/4,\ 1/4)$ or $(4,\ 4)$. Then by Lemma 9, we find a further reduction of ζ with a smaller number of intervals, contradicting the assumption on M. Hence $M \leq 3$. ∎

THEOREM 8: *For any bounded closed interval $J = [a, b]$ with $a < b$, there exists measurable functionals τ: $C(J) \to [0, \infty)$ and G: $C(J) \to \Theta$ such that*

(1) $\Pr[G(Y_J)(t) = \mathbf{N}_{b+t} - \mathbf{N}_b |\ t \leq \tau(Y_J)\ ] = 1$ *for any $t > 0$, and*

(2) $\Pr[\tau(Y_J) < t\ ] \leq 9t/(4B)$ *for any $t > 0$,*

where $C(J)$ is the space of continuous functions on J and we set $B := (b-a)/21$.

Proof: By Theorem 6, there exists a Y_J-measurable synchronized net covering

$[a, b]$. Taking its reduction obtained in Theorem 7, we get a Y_J-measurable synchronized net $\eta := \{\eta_0 < \eta_1 < \cdots < \eta_M\}$ satisfying

(i) $M \leq 3$,

(ii) the synchronized intervals in η have the same level, and

(iii) $\eta_0 \leq a < b \leq \eta_M$.

Define $\tau = \tau(Y_J) := \eta_M - b$ and

$$G(Y_J)(t) := \begin{cases} 0, & t < 0, \\ \mathbf{N}_{b+t} - \mathbf{N}_b, & 0 \leq t \leq \tau, \\ \mathbf{N}_{b+\tau} - \mathbf{N}_b, & t > \tau. \end{cases}$$

Then (1) is clear from the definitions of τ and G together with (3) of Theorem 6.

Let $b \in [\eta_i,\ \eta_{i+1}]$. Then

$$\eta_{i+1} - \eta_i \geq (\eta_M - \eta_0)/(1 + 4 + 4^2) \geq (b-a)/21 = B.$$

Let $[u, v]$ be the minimal synchronized interval containing b with $v - u \geq B$. Since $[u, v] \subset [\eta_i,\ \eta_{i+1}]$, we have $\tau' := v - b \leq \eta_{i+1} - b \leq \eta_M - b = \tau$.

Take $t > 0$ with $t \leq (4/9)B$ and let $n = [B/t]$. If $\tau'(\omega) \in [0, t)$, then $\tau'(\omega - jt) \in [jt, (j+1)t)$ for any $j = 0, 1, \ldots, n-1$. Hence, for any $j = 0, 1, \ldots, n-1$, we have

$$\Pr(\tau'(\omega) \in [0, t)) \leq \Pr(\tau'(\omega - jt) \in [jt, (j+1)t)) = \Pr(\tau'(\omega) \in [jt, (j+1)t)),$$

where we used the fact that the probability measure P is invariant under the addition. Therefore, we have $\Pr(\tau' < t) \leq 1/n$, since

$$n \Pr(\tau' \in [0, t)) \leq \sum_{i=0}^{n-1} P(\tau' \in [jt, (j+1)t)) \leq \Pr(\tau' \in [0, B)) \leq 1.$$

Thus we have (2), since $\Pr(\tau < t) \leq \Pr(\tau' < t) \leq 1/n \leq 9t/(4B)$ for any $t < 4B/9$. For $t \geq 4B/9$, (2) holds trivially since $9t/(4B) \geq 1$. ∎

We construct a predictor for Y_c with $c > b$ based on the observation Y_J, where $J = [a, b]$. We use $G(Y_J)(c)$ to estimate $\mathbf{N}_c - \mathbf{N}_b$. By Theorem 8, if $c - b \leq \tau(Y_J)$, then the estimation is exact. To estimate $Y_c = H(\mathbf{N}_c, c)$, we use the Taylor expansion at $(\mathbf{N}_b,\ b)$ with $G(Y_J)(c)$ for $\mathbf{N}_c - \mathbf{N}_b$:

$$\hat{Y}_c := Y_b + H_x(\mathbf{N}_b, b) G(Y_J)(c) + \frac{1}{2} H_{xx}(\mathbf{N}_b, b) G(Y_J)(c)^2 + H_s(\mathbf{N}_b, b)(c - b).$$

Note that $\hat{Y}_c$ is a measurable function of the observation Y_J by Theorem 6. The value can be calculated based on the observation without using any further information on the unknown function H than (H1) and (H2).

THEOREM 9: *We have*

$$\mathrm{E}[(\hat{Y}_c - Y_c)^2] = o((c-b)^2) + O\Big(\frac{(c-b)^2}{b-a}\Big)$$

as $c \downarrow b$ *with* $C(b)$ *in* (2) *in Section 1 as the constant in* $O(\ )$.

Proof: Since

$$\begin{aligned} Y_c = & Y_b + H_x(\mathbf{N}_b, b)G(Y_J)(c) \\ & + \frac{1}{2}H_{xx}(\mathbf{N}_b, b)G(Y_J)(c)^2 + H_s(\mathbf{N}_b, b)(c-b) + o(c-b), \end{aligned}$$

$\hat{Y}_c - Y_c = o(c-b)$ holds if $c - b \le \tau(Y_J)$. If otherwise, $\hat{Y}_c - Y_c = O((c-b)^{1/2})$ since $|G(Y_J)(c)| \le (c-b)^{1/2}$, $|\mathbf{N}_c - \mathbf{N}_b| \le (c-b)^{1/2}$ and

$$|G(Y_J)(c) - (\mathbf{N}_c - \mathbf{N}_b)| = |\mathbf{N}_{\tau(Y_J)} - \mathbf{N}_b| \le (c-b)^{1/2},$$

so that

$$(\hat{Y}_c - Y_c)^2 \le (1+\delta) \sup_{|x| \le |b|^{1/2}} |H_x(x,b)|^2 (c-b)$$

for any $\delta > 0$ as $c \to b$. Since by Theorem 8, $\Pr[\tau(Y_J) < c-b] \le 48(c-b)/(b-a)$, we have

$$\begin{aligned} \mathrm{E}[(\hat{Y}_c - Y_c)^2] = & \mathrm{E}[(\hat{Y}_c - Y_c)^2 \ |\tau(Y_J) \ge c-b] \ \Pr[\tau(Y_J) \ge c-b] \\ & + \mathrm{E}[(\hat{Y}_c - Y_c)^2 \ |\tau(Y_J) < c-b] \ \Pr[\tau(Y_J) < c-b] \\ \le & o(c-b)^2 + O\Big(\frac{(c-b)^2}{b-a}\Big) \end{aligned}$$

with $C(b)$ in (2) as the constant in $O(\ )$. ∎

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